

# Bifurcation and numerical study in an EHD convection problem

Ioana Dragomirescu

Dept. of Math., Univ. "Politehnica" of Timisoara,  
Piata Victoriei, No.2, 300006, Timisoara, Romania  
i.dragomirescu@gmail.com

## Abstract

The linear eigenvalue problem governing the stability of the mechanical equilibrium of the fluid in a electrohydrodynamic (EHD) convection problem is investigated. The analytical study is one of bifurcation. This allows us to regain the expression of the neutral surface in the classical case. The method used in the numerical study is a Galerkin type spectral method based on polynomials and it provides good results.

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## 1 The physical problem

The investigated physical model is one of the two EHD models of Roberts [9], based on the Gross' experiments [6] which are concerned with a layer of insulating oil confined between two horizontal conducting planes and heated from above and cooled from below. The experimental investigations showed that the presence of a vertical electric field of sufficient strength across the layer, lead to a tessellated pattern of motions, in a manner similar to that of the classical Bénard convection [12]. Gross [6] suggested that this phenomenon may be due to the variation of the dielectric constant of the fluid with the temperature.

In the first model investigated by Roberts [9] the dielectric constant is allowed to vary with the temperature. The homogeneous insulating fluid is assumed to be situated in a layer of depth  $d$  (the fluid occupies the region between the planes  $z = \pm 0.5d$ , which are maintained at uniform but different temperatures), with vertical, parallel applied gradients of temperature and electrostatic potential. The uniform electric field is applied in the  $z$  direction.

The equations governing the EHD convection upon normal mode representations are [12]

$$(D^2 - a^2 - s)(D^2 - a^2 - Prs)(D^2 - a^2)^2 F = La^4 F - Ra a^2 (D^2 - a^2) F \quad (1)$$

with the boundary conditions on  $F$

$$\begin{aligned} F = D^2 F = D(D^2 - a^2 - Prs)F = 0 \\ ((D^2 - a^2)(D^2 - a^2 - s)(D^2 - a^2 - Prs) + R_a a^2)(DF \pm kaF) = 0 \end{aligned} \quad \text{at } z = \pm 0.5. \quad (2)$$

Here, the unknown function  $F$  is the amplitude of the temperature perturbation  $\Theta$ , i. e.  $\Theta = F(z)e^{i(lx+my)+st}$ ,  $Pr$  is the Prandtl number,  $k = \frac{\epsilon_m}{\hat{\epsilon}}$ , with  $\epsilon_m$  the value of the dielectric field at the temperature  $T_m = 0$  and  $\hat{\epsilon}$  the electric constant of the solid in  $z > \frac{1}{2}$ ,  $a$  is the wavenumber,  $a^2 = l^2 + m^2$ ,  $R_a$  is the Rayleigh number,  $L$  is a parameter measuring the potential difference between the planes.

Roberts [9] investigated only the stationary case, i.e.  $s = 0$ , so the eigenvalue problem consists from an eight-order differential equation

$$(D^2 - a^2)^4 F - La^4 F + R_a a^2 (D^2 - a^2) F = 0 \quad (3)$$

and the boundary conditions

$$F = D^2 F = D(D^2 - a^2)F = ((D^2 - a^2)^3 + R_a a^2)(DF \pm kaF) = 0 \text{ at } z = \pm 0.5. \quad (4)$$

He found that when the smallest Rayleigh number,  $R_{a_{\min}} = \min_a R_a(a)$ , varies from  $-1000$  to  $1707.762$ ,  $L$  decreases from  $3370.077$  to  $0$ .

The second model [9] was also been investigated by Turnbull [13], [14]. In this case, the variation of the dielectric constant is not important but the fluid is weakly conducting and its conductivity varies with temperature.

The eigenvalue equation has the form [12]

$$(D^2 - a^2)^3 F + R_a a^2 F + Ma^2 DF = 0 \quad (5)$$

with  $M$  a dimensionless parameter measuring the variation of the electrical conductivity with temperature. The boundary conditions, written for the case of rigid boundaries at constant temperatures, read

$$F = D^2 F = D(D^2 - a^2)F = 0 \text{ at } z = \pm 0.5 \quad (6)$$

Roberts [9] found that when  $M$  is increasing from  $0$  to  $1000$ ,  $R_{a_{\min}}$  is increasing from  $1707.062$  to  $2065.034$ .

Straughan [12] also investigated these EHD convection problems, developing a fully nonlinear energy stability analysis for non-isothermal convection problems in a dielectric fluid.

## 2 The bifurcation analysis

The linear stability of the motion or the equilibrium of a fluid in many problem from hydrodynamic, electrohydrodynamic or hydromagnetic stability theory is governed by a linear higher-order ordinary differential equation

with constant coefficients and homogeneous boundary conditions. The exact solution of such equations or, for the case of eigenvalue problems, the exact eigenvalue is most of the times impossible to find. That is why, numerical methods, usually implying an infinite number of terms, leading however to an approximative solution by some specific truncations to a finite number of terms, are used. However, the theoretical methods can impose restrictions with regards to the numerical results.

For the considered problem let us introduce the direct method [3] which consists in the determination of the eigenfunctions and their introduction into the boundary conditions.

The characteristic equation associated to (5) is

$$(\lambda^2 - a^2)^3 + Ma^2\lambda + R_a a^2 = 0 \quad (7)$$

When the characteristic equation has multiple roots the straightforward application of numerical method can lead to false secular points. That is why, these cases must be investigated separately.

**Proposition 1.** *For  $M = 0$  the only secular points are those situated on*

$$NS_n : R_a = \frac{((2n-1)^2\pi^2 + a^2)^3}{a^2}, \quad \forall n \in \mathbb{N}.$$

*Proof.* For  $M = 0$ , the characteristic equation (7) reduces to

$$(\lambda^2 - a^2)^3 + a_2 = 0, \quad \text{with } a_2 = R_a a^2. \quad (8)$$

In this classical case the roots of (8) have the form

$$\lambda_{1,2} = \sqrt{a^2 + \sqrt[3]{-a_2\epsilon_{1,2}}}, \quad \lambda_3 = \sqrt{a^2 + \sqrt[3]{-a_2}},$$

$$\lambda_4 = -\lambda_1, \lambda_5 = -\lambda_2, \lambda_6 = -\lambda_3$$

so the general solution of (5) has the form

$$F = \sum_{i=1}^3 A_i \cosh(\lambda_i z) + B_i \sinh(\lambda_i z).$$

Replacing the solution  $F$  into the boundary conditions (6) we get the secular equation

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & m_1 & m_2 & m_3 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 m_1 & \lambda_2^2 m_2 & \lambda_3^2 m_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & 0 & 0 & 0 \\ -\lambda_1 \mu_1 m_1 & -\lambda_2 \mu_2 m_2 & -\lambda_3 \mu_3 m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \mu_1 & \lambda_2 \mu_2 & \lambda_3 \mu_3 \end{vmatrix} = 0 \quad (9)$$

with  $m_i = \tanh(\lambda_i/2)$ ,  $\mu_i = \lambda_i^2 - a^2$ ,  $i = 1, 2, 3$ .

When  $\cosh(\lambda_i/2) \neq 0$ ,  $i = 1, 2, 3$ , we can rewrite the secular equation as  $\Delta = \Delta_1 \cdot \Delta_2$  with

$$\Delta_1 = \lambda_1 \mu_1 m_1 (\lambda_2^2 - \lambda_3^2) + \lambda_2 \mu_2 m_2 (\lambda_3^2 - \lambda_1^2) + \lambda_3 \mu_3 m_3 (\lambda_1^2 - \lambda_2^2)$$

and

$$\Delta_2 = \lambda_1^2 m_1 (\lambda_3 \mu_3 m_2 - \lambda_2 \mu_2 m_2) + \lambda_2^2 m_2 (\lambda_1 \mu_1 m_3 - \lambda_3 \mu_3 m_1) + \lambda_3^2 m_3 (\lambda_2 \mu_2 m_1 - \lambda_1 \mu_1 m_2).$$

For  $a > 0$ , the equations  $\Delta_1 = 0$  and  $\Delta_2 = 0$  have only null solutions  $R = 0$ , so no secular points exists on these surfaces.

The condition  $\cosh(\lambda_i/2) \neq 0$ ,  $i = 1, 2, 3$  is not fulfilled only for  $i = 3$ , i.e.  $\cosh(\lambda_3/2) = 0 \Leftrightarrow \cos(\lambda_3/2) = 0 \Leftrightarrow \lambda_3^2 = -(2n-1)^2 \pi^2$ , which implies that the secular curve is  $NS_n$ . And, indeed, the critical values of the Rayleigh number  $R_a$  belong to  $NS_1$  only, identical to the classical one from Chandrasekhar [?].  $\square$

The general form of the solution of the two-point problem for the governing differential equation is written in terms of the roots of the characteristic equation associated with the differential equation. In addition, this form depends on the multiplicity of the characteristic roots. Introducing the general solution into the boundary conditions the secular equation is obtained and it depends on the multiplicity of the characteristic roots. As a consequence, the secular equation has different forms in different regions of the parameter space. Each eigenvalue is a solution of the obtained secular equation, so the eigenvalue depends on all other physical parameters. The neutral manifolds (the most convenient manifolds from the physical point of view), generated by the secular equation separate the domain of stability from the domain of instability.

Let us consider the general case when the roots of the characteristic equation  $\lambda_1, \lambda_2, \dots, \lambda_6$  are distinct. Then the general solution of (5) has the form  $F(z) = \sum_{i=1}^6 A_i e^{\lambda_i z}$ . Introducing it into the boundary conditions (6) we obtain the secular equation [1]

$$\Delta(a, M, R_a) = 0, \quad (10)$$

where  $\Delta$  is a determinant. Its  $i$ -th column has the same form in  $\lambda_i$  as any other  $j$ -th column in  $\lambda_j$ . If  $\lambda_i = \lambda_j$ , then the  $i$ -th and the  $j$ -th columns in  $\Delta$  are identical. Therefore  $\Delta \equiv 0$ . In fact, in this situation, (10) is not entitled to serve as a secular equation and the direct numerical computations will be invalid. When  $M \neq 0$ , some particular cases interesting from the bifurcation point of view, arise due to the existence of bifurcation sets of the characteristic manifold.

Let us consider the surface  $S_0$  defined by the points  $(a, M, R_a) = (a, M, a^4)$ . In this case we have the following result

**Proposition 2.** *Let us define the surfaces*

$$S_i : M = \frac{(33 \mp 3\sqrt{21})\sqrt{90 \pm 10\sqrt{21}a^3}}{250}, \quad i = 1, 2.$$

The surface  $S_0 \cap S_i$ ,  $i = 1, 2$  is a bifurcation set of the characteristic manifold defined by (7). The points on  $S_0 \cap S_i$ ,  $i = 1, 2$  are not secular.

*Proof.* If  $(a, M, R_a) \in S_0$  then  $R_a = a^4$  and one of the roots of the characteristic equation is, for instance,  $\lambda_1 = 0$ . Assuming that  $M \neq 0$  and  $a > 0$ ,  $\lambda_1$  is not a double root of (7). The search of multiple roots reduces then to the equation

$$\lambda^5 - 3\lambda^4 a^2 + 3\lambda^2 a^4 + Ma^2 = 0. \quad (11)$$

No multiple roots of algebraic multiplicity order greater than 2 exists. The double roots of (11) must also be roots of its derivative  $5z^4 - 9z^2 a^2 + 3a^4 = 0$ . In these conditions the possible double roots are  $\lambda_{2,3} = \lambda = -\frac{1}{10}\sqrt{90 \pm 10\sqrt{21}}a$  only for  $(a, M, R_a) \in S_i$ ,  $i = 1, 2$ , i.e. the surfaces  $S_0 \cap S_i$ ,  $i = 1, 2$  are bifurcation sets for the characteristic equation (7). In the case of multiple roots, the general form of the solution of (5) is  $F(z) = \sum_{i=1}^n P_i(z)e^{\lambda_i z}$ , where  $P_i$  is an algebraic polynomial of  $m_i - 1$  degree,  $m_i$  being the algebraic multiplicity of  $\lambda_i$ , in our case  $F(z) = A + (B + Cz)e^{\lambda z} + \sum_{i=4}^6 A_i e^{\lambda_i z}$ .

Formally, the secular equation is deduced from (10), by writing the column  $i$  for  $\lambda_i$  while the columns  $i+1, i+2, \dots, i+m_i-1$  are obtained by differentiating  $l$ ,  $l = 1, 2, \dots, m_i - 1$  times the  $i + l$ -th column of (10) with respect to  $\lambda_{i+l}$  and then replacing  $\lambda_{i+l}$  by  $\lambda_i$ .

However, the numerical evaluations show that now secular points exists on these surfaces. □

### 3 Spectral methods based study

The second part of our study regards the numerical treatment of the two-point problem (5) - (6).

Large classes of eigenvalue problems can be solved numerically using spectral methods, where, typically, the various unknown fields are expanded upon sets of orthogonal polynomials or functions. The convergence of such methods is in most cases easy to assure and they are efficient, accurate and fast. Our numerical study is performed using a weighted residual (Galerkin type) spectral method.

Introducing the new function  $U = (D^2 - a^2)F$ , the generalized eigenvalue problem

$$\begin{cases} (D^2 - a^2)^2 U = -R_a a^2 F - Ma^2 DF, \\ (D^2 - a^2)F = U, \\ F = U = DU = 0 \text{ at } z = \pm 0.5. \end{cases} \quad (12)$$

is obtained. Following [7], we consider the orthogonal sets of functions

$$\{\phi_i\}_{i=1,2,\dots,N} : \phi_i(z) = \int_{-0.5}^z L_i^*(t) dt, \text{ verifying } \phi_i(\pm 0.5) = 0,$$

$$\{\beta_i\}_{i=1,2,\dots,N} : \beta_i(z) = \int_{-0.5}^z \int_{-0.5}^s L_i^*(t) dt ds, \text{ verifying } \beta_i(\pm 0.5) = D\beta_i(\pm 0.5) = 0,$$

with  $L_i^* = L_i(2x)$  the shifted Legendre polynomials on  $(-0.5, 0.5)$  and  $L_i$  the Legendre polynomials on  $(-1, 1)$ . The unknown functions from (12),  $U, F$ , are written as truncated series of functions  $\beta_i$ , irrespective  $\phi_i$ , i.e.

$$U = \sum_{i=1}^N U_i \beta_i(z), \quad F = \sum_{i=1}^N F_i \phi_i(z).$$

The boundary conditions on  $U$  and  $F$  are automatically satisfied. Replacing these expressions in (12), imposing the condition of orthogonality on the vector  $(\beta_k, \phi_k)^T$ ,  $k = 1, 2, \dots, N$ , we get an algebraic system in the unknown, but not all null, coefficients  $U_i, F_i$ . The secular equation, written as the determinant of the obtained algebraic system, gives us the values of the Rayleigh number as a function of the other physical parameters. The smallest values of the Rayleigh number for various values of the parameters  $a$  and  $M$  form the neutral surface that separates the domain of stability from the instability domain. All the expression of the scalar products resulting in the algebraic system are given in [2] for the case of shifted Legendre polynomials on  $(0, 1)$ , but they are easy to adjust to the interval  $(-0.5, 0.5)$ . The specific choice of basis functions led to sparse matrices, with banded sub-matrices of dimension  $N \times N$ .

The numerical evaluations of the critical Rayleigh numbers were obtained for a small number of terms  $N$  ( $N = 6$ ) in the truncated series confirming the well-known accuracy of spectral methods. We obtained that critical values of  $R_a$  are increasing from 1734.120 to 2082.808 when  $M$  is increasing from 0 to 1000, similar to the ones of Roberts [9].

The unknown vector fields from (12) can also be expanded upon complete sequences of functions in  $L^2(-0.5, 0.5)$  defined by using Chebyshev polynomials that satisfy the boundary conditions of the problem. Keeping the above notations, the functions  $\phi_i$ ,  $i = 1, 2, \dots, N$  are defined by  $\phi_i(z) = T_i^*(z) - T_{i+2}^*(z)$  and  $\beta_i$ ,  $i = 1, 2, \dots, N$  by  $\beta_i(z) = T_i^* - \frac{2(i+2)}{i+3} T_{i+2}^* + \frac{i+1}{i+3} T_{i+4}^*$  [11] with  $T_i^*$ ,  $i = 1, 2, \dots, N$ , the shifted Chebyshev polynomials on  $(-0.5, 0.5)$  defined in a similar manner as the shifted Legendre polynomials. All the evaluations of the scalar products were based on the orthogonality relation

$$\int_{-0.5}^{0.5} T_n^*(z) T_m^*(z) w^*(z) dz = \begin{cases} \frac{\pi}{2} c_n \delta_{nm}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (13)$$

with respect to the weight function  $w^*(z) = \frac{1}{\sqrt{1/4 - z^2}}$ .

The numerical results where obtained for a larger number of terms in the expansion sets ( $N = 11$ ) and they show that the shifted Legendre based method is more effective in this case. We can mention that the Chebyshev polynomials are considered suitable more likely for the tau method or the collocation type methods. Some numerical evaluations of  $R_a$  as a function of  $a$  and  $M$  are given in Table 1.

Other sets of complete orthogonal functions based on Chebyshev polynomials and satisfying various boundary conditions can be found in [5], [8].

$a$	$M$	$R_a - SLP$	$R_a - SCP$
3.117	0	1734.120	1775.955
3.117	10	1734.154	1775.987
3.117	1000	2082.802	2100.935
1.5	0	3116.286	31199.286
1.5	5	3116.381	3199.289
10	0	11409.157	14909.559
10	100	14414.05	14419.963
10	500	14531.694	14994.747
20	0	166779.036	182878.881

Table 1: Numerical values for the Rayleigh number for various values of the parameters  $a$ ,  $M$  obtained by spectral methods based on shifted Legendre (SLP) and shifted Chebyshev (SCP) polynomials.

## 4 Conclusions

In this paper we performed a bifurcation analysis and a numerical treatment for an electrohydrodynamic convection problem. When eigenvalue problems from linear stability theory are investigated only numerically spurious eigenvalue can be encountered, especially when bifurcation sets of the characteristic manifold occur. In order to detect the false secular points a bifurcation study of the problem becomes necessary. An example of this type of problems was investigated in [4], e.g. for an electrohydrodynamic convection problem in the case of free-free boundaries the numerical methods led to the existence of false secular points.

The numerical study was performed here using a Galerkin type spectral method which implied that the boundary conditions are satisfied by the orthogonal sets of expansion functions. However, when this condition is not fulfilled, the tau method or the collocation method can also be applied. All these methods are widely used in the numerical investigation of eigenvalue problems governing the linear stability of motions or equilibrium of fluid in convection problems. From the physical point of view, the evaluations of the Rayleigh number  $R_a$  showed an enlargement of the stability domain when the parameter  $M$  is increasing, the dependence of  $R_a$  of  $M$  is not however exponential. These evaluations were easy to compute and proved to be similar to the ones existing in the literature.

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